# Toward the Collapse of State

Slobodan Prvanović and Zvonko Marić
Institute of Physics, P.O. Box 57, 11001 Belgrade,
Serbia

February 1, 2008

**Abstract**

The basic concepts of classical mechanics are given in the operator form. Then, the hybrid systems approach, with the operator formulation of both quantum and classical sector, is applied to the case of an ideal nonselective measurement. It is found that the dynamical equation, consisting of the Schrödinger and Liouville dynamics, produces noncausal evolution when the initial state of measured system and measuring apparatus is chosen to be as it is demanded in discussions regarding the problem of measurement. Nonuniqueness of possible realizations of transition from pure noncorrelated to mixed correlated state is analyzed in details. It is concluded that collapse of state is the only possible way of evolution of physical systems in this case.

## 1 Introduction

The correct theory of combined quantum mechanical and classical mechanical systems has to differ from quantum mechanics (QM) and classical mechanics (CM) with respect to causality and related topics. This is because the dynamical equations of QM and CM, taken separately, cannot lead to such changes of states that can happen in a (quantum) measurement processes. Quantum and classical mechanics are causal theories in which pure states can evolve, according to the appropriate equations of motion, only into pure states, not into the mixed ones. For a process of nonselective measurement on some QM system, done by an apparatus which is a CM system, there is a possibility for transitions from pure to mixed state.

An interesting approach to hybrid systems (consisting of one quantum and one classical system) was proposed in literature [1, 2]. In short, it uses for states and observables the direct product of QM and CM representatives. The dynamical equation there introduced is, let say, superposition of QM and CM dynamical equations. But, it was objected that this equation of motion does not save the non-negativity of states, which has to be unaltered if the theory is supposed to be physically meaningful. Otherwise, there would be events whose occurrence is characterized with negative probabilities. However, we shall try to show that the hybrid systems approach (HSA) is the adequate theoretical framework for description of an ideal nonselective measurement.

The employed strategy will be the following. Firstly, for a particular choice of initial state of QM system and measuring apparatus, which addresses the problem of measurement, it will be shown that a correlated state, in contrast to the initial, cannot be pure. Secondly, it will be found that the (dramatic) change of purity can be formally realized in more than one way; only one of them will be unphysical for involved negative probabilities. In order to find what should be taken as the state of this hybrid system after the beginning of measurement, the subtle analysis is needed. It should support ones belief that the change of purity is necessarily followed by the change of this or that property of state.

We shall keep the argumentation on the physical ground. Precisely, the necessary requirements to respect the physical meaning whenever it is possible, and/or to consider only physically meaningful mathematical entities when physical problems are discussed, will be sufficient here for finding the other, physically meaningful possibility for mixed correlated state as the result. It will become obvious that this state is in accordance with expected collapse of QM state, as is suggested by the above (anarchical) title.

Before showing that, we shall propose an operator formulation of classical mechanics. We shall use it instead of the standard phase space formulation of CM within the HSA. It will allow us to proceed the argumentation in more complete way. However, it can be used separately with some other intentions.

## 2 The Operator Description of Classical Mechanics

The most important features of the well-known phase space formulation of classical mechanics are: **1.)** the algebra of observables is commutative, **2.)** the equation of motion is the Liouville equation and it incorporates the Poisson bracket and **3.)** pure states are those with sharp values of position and momentum, the values of which are, in general, independent. All these will hold for the operator formulation of CM which we are going to introduce heuristically.

Let the pure states for position, in the Dirac notation, be $|q\rangle$. Similarly, for momentum: $|p\rangle$. In quantum mechanics independence of states is formalized by the use of direct product. These prescriptions suggest that pure classical states should be related somehow with $|q\rangle \otimes |p\rangle$. Consequently, the operator formulation of classical mechanics should be looked for within the direct product of two rigged Hilbert spaces, let say $\mathcal{H}^q \otimes \mathcal{H}^p$. In such a space, one can define an algebra of classical observables. It is the algebra of polynomials in $\hat{q}_{cm} = \hat{q} \otimes \hat{I}$ and $\hat{p}_{cm} = \hat{I} \otimes \hat{p}$ with real coefficients, *etc.* The elements of this algebra are Hermitian operators and they obviously commute since $[\hat{q}_{cm}, \hat{p}_{cm}] = 0$. Further, one can define states like in the standard formulation of CM as functions of position and momentum, which are now operators. Precisely, one can define the pure states as:

$$\delta(\hat{q} - q(t)) \otimes \delta(\hat{p} - p(t)) = \int\int \delta(q - q(t))\delta(p - p(t))|q\rangle\langle q| \otimes |p\rangle\langle p| dqdp =$$
$$= |q(t)\rangle\langle q(t)| \otimes |p(t)\rangle\langle p(t)|. \tag{1}$$

The pure and (noncoherently) mixed states, commonly denoted by $\rho(\hat{q}_{cm}, \hat{p}_{cm}, t)$ in this formulation, are non-negative and Hermitian operators, normalized to $\delta^2(0)$ if for the same function of real numbers, *i.e.*, for $\rho(q, p, t)$, it holds that $\rho(q, p, t) \in \mathbf{R}$, $\rho(q, p, t) \geq 0$ and $\int\int \rho(q, p, t)\ dq\ dp = 1$. If one calculates the mean values of observables, *e.g.*, $f(\hat{q}_{cm}, \hat{p}_{cm})$, in state $\rho(\hat{q}_{cm}, \hat{p}_{cm}, t))$ by the Ansatz:

$$\frac{\mathrm{Tr}(f(\hat{q}_{cm}, \hat{p}_{cm})\rho(\hat{q}_{cm}, \hat{p}_{cm}, t))}{\mathrm{Tr}\rho(\hat{q}_{cm}, \hat{p}_{cm}, t)}, \tag{2}$$

then it will be equal to the usually calculated $\int\int f(q, p)\rho(q, p, t) dqdp$ where $f(q, p)$ and $\rho(q, p, t)$ are the phase space representatives of corresponding observable and state, respectively. It is easy to see that, due to (1) and (2), the third characteristic of phase space formulation holds for the new one as well.

For the criterion of purity we propose the idempotency of state, up to its norm. This criterion is obviously satisfied for (1) and it is adequate for the standard formulation of QM. Therefore, we shall use it for the operator formulation of hybrid systems, too.

The dynamical equation in the new formulation can be defined in accordance to **2.)** as:

$$\frac{\partial\rho(\hat{q}_{cm},\hat{p}_{cm},t)}{\partial t}=\frac{\partial H(\hat{q}_{cm},\hat{p}_{cm})}{\partial\hat{q}_{cm}}\frac{\partial\rho(\hat{q}_{cm},\hat{p}_{cm},t)}{\partial\hat{p}_{cm}}-\frac{\partial H(\hat{q}_{cm},\hat{p}_{cm})}{\partial\hat{p}_{cm}}\frac{\partial\rho(\hat{q}_{cm},\hat{p}_{cm},t)}{\partial\hat{q}_{cm}}. \tag{3}$$

For the RHS of (3) we shall use the notation $\{H(\hat{q}_{cm},\hat{p}_{cm}),\rho(\hat{q}_{cm},\hat{p}_{cm},t)\}$.

The standard formulation of classical mechanics appears through the kernels of the operator formulation in the $|q\rangle\otimes|p\rangle$ representation. This, together with (2), can be used as the proof of equivalence of the two formulations. The other important remark is that, after the $\hat{q}_{cm}$ and $\hat{p}_{cm}$ have been defined, each other observable and every state can and have to be expressed as some function of just these two.

# 3 An Outline of the Hybrid Systems Approach

A physical system is called hybrid system if it consists of one QM and one CM system. Such systems were discussed in [1-7]. Instead of reviewing these articles with purpose of introducing formalism for hybrid systems, we shall start with the standard treatment of two QM systems and then, by substituting one quantum with one classical system, find directly the appropriate theoretical framework.

The standard formulation of two quantum systems needs the direct product of two (rigged) Hilbert spaces, let say $\mathcal{H}_{qm1}\otimes\mathcal{H}_{qm2}$. The states of these systems evolve according to the Schrödinger equation with the Hamiltonian

$\sum_\alpha \hat{H}^\alpha_{qm1} \otimes \hat{H}^\alpha_{qm2}$, for which it holds:

$$\frac{\partial(\sum_{ij} \hat{\rho}^{ij}_{qm1}(t) \otimes \hat{\rho}^{ij}_{qm2}(t))}{\partial t} = \frac{1}{i\hbar}[\sum_\alpha \hat{H}^\alpha_{qm1} \otimes \hat{H}^\alpha_{qm2}, \sum_{ij} \hat{\rho}^{ij}_{qm1}(t) \otimes \hat{\rho}^{ij}_{qm2}(t)] =$$

$$= \sum_{\alpha ij} \frac{1}{i\hbar}[\hat{H}^\alpha_{qm1}, \hat{\rho}^{ij}_{qm1}(t)] \otimes \frac{\hat{H}^\alpha_{qm2}\hat{\rho}^{ij}_{qm2}(t) + \hat{\rho}^{ij}_{qm2}(t)\hat{H}^\alpha_{qm2}}{2} +$$

$$+ \sum_{\alpha ij} \frac{\hat{H}^\alpha_{qm1}\hat{\rho}^{ij}_{qm1}(t) + \hat{\rho}^{ij}_{qm1}(t)\hat{H}^\alpha_{qm1}}{2} \otimes \frac{1}{i\hbar}[\hat{H}^\alpha_{qm2}, \hat{\rho}^{ij}_{qm2}(t)]. \tag{4}$$

With $\sum_{ij} \hat{\rho}^{ij}_{qm1}(t) \otimes \hat{\rho}^{ij}_{qm2}(t)$ (and more with the one in next expression) we want to accommodate the notation for states to that type of correlation which will be discussed below.

Suppose now that the second system is classical. This means that everything related to this system in (4) has to be translated into the classical counterparts. Having in mind the above formulation of CM, we propose:

$$\frac{\partial(\sum_{ij} \hat{\rho}^{ij}_{qm}(t) \otimes \hat{\rho}^{ij}_{cm}(t))}{\partial t} =$$

$$= \sum_{\alpha ij} \frac{1}{i\hbar}[\hat{H}^\alpha_{qm}, \hat{\rho}^{ij}_{qm}(t)] \otimes \frac{\hat{H}^\alpha_{cm}\hat{\rho}^{ij}_{cm}(t) + \hat{\rho}^{ij}_{cm}(t)\hat{H}^\alpha_{cm}}{2} +$$

$$+ \sum_{\alpha ij} \frac{\hat{H}^\alpha_{qm}\hat{\rho}^{ij}_{qm}(t) + \hat{\rho}^{ij}_{qm}(t)\hat{H}^\alpha_{qm}}{2} \otimes \{\hat{H}^\alpha_{cm}, \hat{\rho}^{ij}_{cm}(t)\}, \tag{5}$$

as the dynamical equation. Few explanations follow. The first system remained quantum mechanical, so its type of evolution is left unaltered. The Poisson bracket is there instead of the second commutator because classical systems evolve according to the Liouville equation. It is defined as in (3); the partial derivatives are with respect to the classical coordinate and momentum: $\hat{q} \otimes \hat{I}$ and $\hat{I} \otimes \hat{p}$. All states and observables, both QM and CM, appear in the operator form, *i.e.*, hybrid system is defined in $\mathcal{H}_{qm} \otimes \mathcal{H}^q_{cm} \otimes \mathcal{H}^p_{cm}$. (*Nota bene*, the coordinate and momentum of quantum and classical systems are operators acting in $\mathcal{H}_{qm}$ and $\mathcal{H}^q_{cm} \otimes \mathcal{H}^p_{cm}$, respectively.) Some justifications of (5) we shall give in due course.

Similar equations, in the c-number formulation of CM, were proposed in [1-4]. There one can find the whole variety of requests that has to be imposed on the equation of motion for hybrid systems which will not be reviewed here. We just mention that the equation proposed in [1-3] is antisymmetric, while the one in [4] is not.

More discussions of the same subject one can find in [5, 6]. The starting point there was that the formalism of hybrid systems should have all mathematical properties of QM and CM (see [6] for details) and it was concluded that such formalism cannot exist. Rather than as a critique, we understand this result as an indication that the HSA is on a right track. Namely, we do not expect from the appropriate formalism to posses all mathematical properties being the same as in quantum and classical mechanics. On the contrary, we expect that the correct theory of hybrid systems will differ from these two mechanics with respect to the causality of evolution and, consequently, all other related topics. More precisely, in some cases the hybrid systems equation of motion should lead to the noncausal evolution. The example we have in mind, as we have mentioned, is a process of (quantum) measurement.

It was objected in [2, 3, 7] that the HSA dynamical equation does not save the non-negativity of states. Our intention is to show, with a subtle analysis of process of measurement, that this need not to be so, *i.e.*, the non-negativity of states can be saved. This comes from our belief that after finding some dynamical equation as the source of noncausal evolution, what will be the case for (5), one should accept any kind of instruction, of course, if there is some, since, on the first place, one would be faced with the problem in which way it should be solved. That is, this type of dynamical equations, we believe, should be approached in different, more careful manner than it is usually the case because it is not so straitforward job to solve them. On the other hand, it will be enough to apply some arguments, that are of the same kind as are those which qualify non-negative states as meaningless, and to find acceptable states. This will become clear latter. At this place, let us just mention that the noncausal evolution of CM system alone occurred in a treatment of classical mechanics by the inverse Weyl transform of the Wigner function; see [8] for details.

## 4 The Process of Measurement

Usually, it is said that the measuring apparatus is classical system. The formalism of hybrid systems becomes then the natural choice for the representation of process of (quantum) measurement. We shall consider the nonselective measurement within the operator formulation of HSA by taking that the states of measured QM system and measuring apparatus evolve under the action of $H_{qm}(\hat{q}\otimes\hat{I}\otimes\hat{I},\hat{p}\otimes\hat{I}\otimes\hat{I})+H_{cm}(\hat{I}\otimes\hat{q}\otimes\hat{I},\hat{I}\otimes\hat{I}\otimes\hat{p})+V_{qm}(\hat{q}\otimes\hat{I}\otimes\hat{I},\hat{p}\otimes\hat{I}\otimes\hat{I})\cdot V_{cm}(\hat{I}\otimes\hat{q}\otimes\hat{I},\hat{I}\otimes\hat{I}\otimes\hat{p})$. To simplify the expressions, we shall use $\hat{H}_{qm}\otimes\hat{I}_{cm}+\hat{I}_{qm}\otimes\hat{H}_{cm}+\hat{V}_{qm}\otimes\hat{V}_{cm}$ as the notation for this Hamiltonian. The measured observable is $\hat{V}_{qm}=\sum_i v_i|\psi_i\rangle\langle\psi_i|\otimes\hat{I}\otimes\hat{I}$. It is necessary that $[\hat{H}_{qm},\hat{V}_{qm}]=0$ because, if the quantum system before the measurement was in one of the eigenstates of the measured observable, say $|\psi_i\rangle$, then this system would not change its state during the measurement. Then, $\hat{H}_{qm}$ can be diagonalized in the same basis: $\hat{H}_{qm}=\sum_i h_i|\psi_i\rangle\langle\psi_i|\otimes\hat{I}\otimes\hat{I}$. For the CM parts of Hamiltonian it is reasonable to assume that they do not cause periodic motion of the pointer. We shall not specify the Hamiltonian in more details because we are interested only in discussions related to the form of state after the beginning of measurement.

For the initial state of quantum system we shall take the pure state $|\Psi(t_o)\rangle$ and for the pointer of apparatus we shall take that initially it is in the state with sharp values of position and momentum, let say $q_o$ and $p_o$, so the state of hybrid system at the moment when measurement starts is $\hat{\rho}_{qm}(t_o)\otimes\hat{\rho}_{cm}(t_o)=|\Psi(t_o)\rangle\langle\Psi(t_o)|\otimes|q_o\rangle\langle q_o|\otimes|p_o\rangle\langle p_o|$. Of course, the problem of measurement demands $|\Psi(t_o)\rangle$ to be superposition $\sum_i c_i(t_o)|\psi_i\rangle$.

Due to the interaction term in Hamiltonian, the state of composite system will become correlated - the CM parts of state will depend somehow on the eigenvalues of $\hat{V}_{qm}$. Let us use the notation $\sum_{ij}\hat{\rho}^{ij}_{qm}(t)\otimes\hat{\rho}^{ij}_{cm}(t)$ in order to allow the analysis of, *a priori*, possible situation in which the CM parts of state can depend on two different eigenvalues of $\hat{V}_{qm}$. With this notation, and the above for Hamiltonian, the dynamics of measurement becomes represented with:

$$\frac{\partial(\sum_{ij}\hat{\rho}^{ij}_{qm}(t)\otimes\hat{\rho}^{ij}_{cm}(t))}{\partial t}=$$

$$=\sum_{ij}\frac{1}{i\hbar}[\hat{H}_{qm},\hat{\rho}^{ij}_{qm}(t)]\otimes\hat{\rho}^{ij}_{cm}(t)+\sum_{ij}\frac{1}{i\hbar}[\hat{V}_{qm},\hat{\rho}^{ij}_{qm}(t)]\otimes\hat{V}_{cm}\hat{\rho}^{ij}_{cm}(t)+$$

$$+\sum_{ij}\hat{\rho}_{qm}^{ij}(t)\otimes\{\hat{H}_{cm},\hat{\rho}_{cm}^{ij}(t)\}+\sum_{ij}\frac{1}{2}(\hat{V}_{qm}\hat{\rho}_{qm}^{ij}(t)+\hat{\rho}_{qm}^{ij}(t)\hat{V}_{qm})\otimes\{\hat{V}_{cm},\hat{\rho}_{cm}^{ij}(t)\},\tag{6}$$

where $\hat{H}_{cm}, \hat{V}_{cm}$ and $\rho_{cm}^{ij}(t)$ are derived in the Poisson bracket with respect to $\hat{q}\otimes\hat{I}$ and $\hat{I}\otimes\hat{p}$ that act in $\mathcal{H}_{cm}^{q}\otimes\mathcal{H}_{cm}^{p}$.

The solution of this dynamical equation will represent the state of hybrid system at $t > t_o$ and the search for it can start by noticing that the CM terms $\hat{\rho}_{cm}^{ii}(t)$, attached to the quantum mechanical terms with equal indices $\hat{\rho}_{qm}^{ii}(t)$ (which we shall call diagonal terms), are $\hat{\rho}_{cm}^{ii}(t) = |q_i(t)\rangle\langle q_i(t)|\otimes|p_i(t)\rangle\langle p_i(t)|$, where the indices in $|q_i(t)\rangle$ and $|p_i(t)\rangle$ underline dependence on one eigenvalue of $\hat{V}_{qm}$. Being guided by this dependence of each CM bra and ket of $\hat{\rho}_{cm}^{ii}(t)$ on one eigenvalue of $\hat{V}_{qm}$, as the candidate for correlated state we shall consider the coherent mixture:

$$\sum_{ij} c_{ij}(t)|\psi_i\rangle\langle\psi_j|\otimes|q_i(t)\rangle\langle q_j(t)|\otimes|p_i(t)\rangle\langle p_j(t)|.\tag{7}$$

There are two other candidates for correlated state. The first is:

$$\sum_{ij} c_{ij}(t)|\psi_i\rangle\langle\psi_j|\otimes|q_{ij}(t)\rangle\langle q_{ij}(t)|\otimes|p_{ij}(t)\rangle\langle p_{ij}(t)|,\tag{8}$$

where the indices in $|q_{ij}(t)\rangle$ and $|p_{ij}(t)\rangle$ stand to represent dependence on two eigenvalues of $\hat{V}_{qm}$ in the form $\frac{1}{2}(v_i+v_j)$. The same holds for $\langle q_{ij}(t)|$ and $\langle p_{ij}(t)|$. The motivation for this comes from the symmetrization of QM sector in front of the Poisson bracket on the RHS of (5). The terms $\hat{\rho}_{cm}^{ij}(t)$ of (8) are diagonal with respect to the eigenbasis of $\hat{q}_{cm}$ and $\hat{p}_{cm}$ for each pair of indices, while these terms of (7) for $i\neq j$ are not. As the third candidate for correlated state we shall consider the noncoherent mixture:

$$\sum_{i}|c_i(t_o)|^2|\psi_i\rangle\langle\psi_i|\otimes|q_i(t)\rangle\langle q_i(t)|\otimes|p_i(t)\rangle\langle p_i(t)|.\tag{9}$$

All three states have the same diagonal terms $\hat{\rho}_{qm}^{ii}(t)\otimes\hat{\rho}_{cm}^{ii}(t)$. The difference between these states is in the CM $i\neq j$ terms. Each ket and bra of $\hat{\rho}_{cm}^{ij}(t)$, $i\neq j$, in (7) depends on only one eigenvalue of $\hat{V}_{qm}$, in (8) they depend on two eigenvalues and in expression (9) there are no such terms.

The state (7) is designed to represent as pure, non-negative and Hermitian correlated state as is the initial state and it has nondiagonal QM terms (with respect to the basis $|\psi_i\rangle$) as the state $|\Psi(t_o)\rangle\langle\Psi(t_o)|$. (The state is taken to be pure if it is idempotent up to the norm: $\hat{\rho}^2 = \delta^2(0)\cdot\hat{\rho}$.) The purity of (7) rests on the same type of time development (dependence on one $v_i$) of $|q_i(t)\rangle$ and $|p_i(t)\rangle$, no matter do they belong to $\hat{\rho}^{ij}_{cm}(t)$ with $i = j$ or with $i \neq j$. But, the following holds. The initial state of the apparatus is diagonal with respect to the eigenbasis of $\hat{q}_{cm}$ and $\hat{p}_{cm}$. To "create" the nondiagonal terms from it in the form which ensures purity, one would need to introduce operators that do not commute with $\hat{q}_{cm}$ and $\hat{p}_{cm}$ to act on CM states. One would need to take some other dynamical equation instead of (5) as well. That dynamical equation should use commutator for both subsystems, like it is the case for (4). If one would do that, then, in a treatment of the apparatus, one would neglect the requirements **1.)** and **2.)** which are the part of definition of classical systems (see Sec. 2). This type of reasoning would be *a la* von Neumann's approach to measurement process where the apparatus and measured system are both treated as quantum systems. Instead of going in that direction, we are considering here the apparatus as classical system, defined in the above given way. By this we avoid the well known problems that arise with states such is (7). (According to (7) there could be a superposition of pointers state which is unobserved. Then, the problem of measurement, as we understand it, is to explain why and describe how the state similar to (7) collapses to the state similar to (9).)

The less descriptive and more formal way to look for a solution is to assume that the time dependence of evolved state is as represented by (7). Then, by substituting (7) in (6) in order to verify this, we find a contradiction. Namely, the CM $i \neq j$ terms of (7) do not commute with $\hat{q}_{cm}$ and $\hat{p}_{cm}$ for $t \neq t_o$, so then they are not functions of only these observables. The partial derivatives $\frac{\partial}{\partial \hat{q}_{cm}}$ and $\frac{\partial}{\partial \hat{p}_{cm}}$ from the Poisson bracket "annihilate" the CM nondiagonal elements of (7) for $t > t_o$ when they act on them. For instance:

$$\frac{\partial}{\partial \hat{q}}|q_i(t)\rangle\langle q_j(t)| = \frac{\partial}{\partial \hat{q}}\delta(\hat{q} - q_i(t))\cdot\delta_{i,j}, \tag{10}$$

($t > t_o$) and similarly for $|p_i(t)\rangle\langle p_j(t)|$ under the action of $\frac{\partial}{\partial \hat{p}_{cm}}$. Thus, for the CM $i \neq j$ terms of (7) the RHS of (6) vanishes for $t > t_o$, while the LHS is not equal to zero by assumption.

Let us stop for a moment and put few remarks. An immediate consequence of the fact that (7) does not satisfy (6) is that the initial purity of state is lost due to the established correlation. This is confirmed by considerations of (8) and (9). These two states do satisfy (6), but they are both mixed - they are not idempotent up to the norm: $\hat{\rho}^2 \neq \delta^2(0) \cdot \hat{\rho}$. This property is plausible for (9). For (8) it is enough to notice that in $\hat{\rho}^2$ there is, for example, term $|\psi_i\rangle\langle\psi_i| \otimes |q_{ij}(t)\rangle\langle q_{ij}(t)| \otimes |p_{ij}(t)\rangle\langle p_{ij}(t)|$ which is not present in $\hat{\rho}$. Therefore, the hybrid systems dynamical equation produces in this particular case a noncausal evolution: pure noncorrelated state transforms in some mixed correlated state (which is to be found). This is the crucial difference between (5) and the Schrödinger and Liouville dynamics that appear within it.

One can convince oneself, by looking at (8) and (9), that purity is not the only property of initial state that changes instantaneously at the moment when interaction begins. Obviously, there are no $i \neq j$ terms in (9) the meaning of which is that the QM part of (9), in difference to the initial one, is diagonal with respect to the basis $|\psi_i\rangle$. On the other hand, the state (8) is not non-negative operator for all $t > t_o$, while the initial state is. For all states that are not non-negative operators one can construct properties - events, that would be "found" with negative probabilities if they would be measured. In order to construct such a property for (8), it is helpful to notice that the CM parts of $i \neq j$ terms of (8) are regular states of CM systems, they are different from those with $i = j$ and they are accompanied by the QM "states" with vanishing trace. (By regular we mean *per se* realizable since they are diagonal and "states" stands here, and would be better to stay in all similar cases, because they can only be interpreted as impossible.)

For the related negative probabilities, states which are not non-negative operators should be qualified as meaningless and, since they appeared in the HSA, there were objections on its relevance for physics. In what follows, we want to show that these probabilities are not unavoidable here. In other words, our intention is to rehabilitate the HSA and this will manifest itself in finding formal support for physically meaningful state (9), that it should be taken as the solution, not the unphysical state (8). The arguments have to be in accordance with physics since the experience makes one to be unsatisfied with (8) and, of course, the HSA is aimed to formalize behaviour of physical systems. The first argumentation, being based on the validity of (10), will continue the analysis of (7). The second discussion, concentrated on (8) and unrelated to (10), will again designate that (9) is the proper solution, but, in difference to the first one, it will be proceeded in more interpretational than formal manner.

Our insistence on (7) rests on the fact that one can look on it as on a trial state. It is the perfect choice for a trial state because it has the same physically relevant characteristics as the initial state and it is equal to the initial state for $t = t_o$, *i.e.*, for $t \to t_o$ (7) approaches the initial state without any change when these characteristics are considered. Moreover, the need for a trial state comes from the absence (up to our knowledge) of some rule that would prescribe how to manage the change of idempotency. After being substituted on the RHS of dynamical equation, trial state will indicate the appropriate type of time transformation. Then, by minimal modifications of this state, intended to adapt it to that type, desired correlated state will be found.

The RHS of (6) for the CM $i \neq j$ terms of (7) vanishes for all $t > t_o$ according to (10). Exclusively for $t = t_o$ the CM $i \neq j$ terms of (7) can be expressed as functions of only $\hat{q}_{cm}$ and $\hat{p}_{cm}$ since $q_i(t_o) = q_o$ and $p_i(t_o) = p_o$ for all $i$. Only for this moment the RHS of (6) for CM $i \neq j$ terms of (7) does not vanish. Therefore, one concludes that the CM parts of $i \neq j$ terms has to be constant after the instantaneous change at $t_o$, *i.e.*, instead with those of (7), the QM nondiagonal terms have to be coupled with the time independent CM terms for all $t > t_o$. This is how dynamical equation designates that (8) should not be taken as the solution. What one has to do, if one wants to accommodate (7) to deduced time independence of the CM $i \neq j$ terms, is to take for these terms (for $t > t_o$) some operators that do not involve time. Then, in order to satisfy (6), that operators should not be expressible as some functions of (only available) $\hat{q}_{cm}$ and $\hat{p}_{cm}$. On the other hand, with these operators one should not change neither the Hermitian character nor the non-negativity of state since nothing asks that. The resulting state, of course, has to be impure because any change of the CM $i \neq j$ terms of (7) affects its idempotency. In this way, (9) will be obtained as the appropriate solution.

Having in mind the functions of $\hat{q}_{cm}$, $\hat{p}_{cm}$ and operators that do not commute with these two, one may want not to accept (10). For the sake of mathematical rigor, let us clear up this. The CM nondiagonal terms of (7) cannot be expressed as some functions depending only on $\hat{q}_{cm}$ and $\hat{p}_{cm}$, but they can be expressed as some functions of these two if, firstly, the number of the operators available is increased and, secondly, there is some noncommutativity among them. How this functions would look like depends on these new operators. Since there are neither motivations nor instructions for their introduction coming from physics, they can be introduced liberately. More precisely, these operators do not represent anything meaningful and they need not to enclose any known mathematical structure. For instance, $|q_i(t)\rangle\langle q_j(t)|$ can be expressed as $\exp(\frac{1}{a}(q_i(t)-q_j(t))\hat{\pi})\delta(\hat{q}-q_j(t))$, where $\hat{\pi}$ is not to be confused with the CM momentum since it acts in $\mathcal{H}^q$, not in $\mathcal{H}^p$, and $\langle q|\hat{\pi}|q'\rangle = a\frac{\partial\delta(q-q')}{\partial q}$. Here, $a$ can be anything, it need not to be equal to $-i\hbar$ as in quantum mechanics. The other (even more pathological) example is the following. Since the CM nondiagonal dyads do not commute with $\hat{q}_{cm}$ and $\hat{p}_{cm}$, they can be used as the new operators, *e.g.*, $|q_i(t)\rangle\langle q_j(t)| = F(q_i(t))^{-1}F(\hat{q})|q_i(t)\rangle\langle q_j(t)|$. This shows that these nondiagonal dyads can be expressed as functions depending on $\hat{q}_{cm}$, $\hat{p}_{cm}$ and uncountably many other arguments - all nondiagonal dyads, where $F$ can be any function. With these two examples we wanted to justify the need to bound considerations of CM in operator form to functions of only $\hat{q}_{cm}$ and $\hat{p}_{cm}$. On the other hand, the request to discuss purity of state of the hybrid system has risen the need to consider nondiagonality (with respect to the basis $|q\rangle\otimes|p\rangle$) of CM state. When these two meet in dynamical equation, with expressions like (10) nothing unusual was done: the derivation of an entity, which is not some function of that with respect to which it is derived, has zero as a result. If one says that the LHS of (10) is just defined by the RHS of (10), then it should be noticed that (10) does not contradict any of the calculational rules of CM and QM because in the standard formulation of classical mechanics there is no possibility for realization of nondiagonality, while in the standard formulation of quantum mechanics there is no necessity for restriction to commutativity. Anyhow, let us proceed by supposing that one is not willing to accept (10) and/or that one finds the given support for (9) as not enough convincing.

Even without (10), one is not free of contradiction if (7) is taken to be the solution. Due to the symmetrization of QM sector, on the RHS of (6), in front of the second Poisson bracket, there are two eigenvalues of $\hat{V}_{qm}$ coming from $\hat{\rho}^{ij}_{qm}(t)$ $(i \neq j)$ of (7). Because of this, the assumption that each ket and bra of $\hat{\rho}^{ij}_{cm}(t)$ $(i \neq j)$ of (7) depends on only one eigenvalue of $\hat{V}_{qm}$ is contradicted. As it seems, to introduce non-commuting operators in $\mathcal{H}^q_{cm}\otimes\mathcal{H}^p_{cm}$, and/or to slightly modify (6), would not be enough to avoid some contradiction connected to (7) when it is seen as the result of evolution. However, it is not our intention to go in these directions because it would be against the purpose of this article.

After discarding (7), one concludes that each ket and bra of $\hat{\rho}^{ij}_{cm}(t)$ $(i \neq j)$ would depend on two eigenvalues of $\hat{V}_{qm}$ coming from $\hat{\rho}^{ij}_{qm}(t)$ $(i \neq j)$ for $t > t_o$ if there would be $\hat{\rho}^{ij}_{qm}(t)$ $(i \neq j)$ for that times at all. Therefore, the most important step in solving dynamical equation for the above Hamiltonian is to find what happens with the initial QM state at the moment when interaction begins. Then it will be almost trivial problem to find the state of hybrid system at latter times. Or, more precisely, in the presence of $|\psi_i\rangle\langle\psi_j|$ $(i \neq j)$ for $t > t_o$ is the origin of dilemma: (8) or (9), the meaning of which is that by the assumed linearity of evolution, in a case when it is noncausal, one excludes the physical meaning of evolved state, and *vice versa*.

From this point, our strategy for defense of the HSA from objections that it might be unphysical is in showing that one finds it unphysical only after one has previously decided to prefer formal, rather than physical arguments and, moreover, only after one has neglected statements (being, by the way, of the same sort as those used for disqualification) that lead to physically meaningful state. Let us be more concrete. To find (8) it was necessary to start with more formal assumption that the nondiagonality of QM part of state, with respect to the eigenbasis of $\hat{H}_{qm}$ and $\hat{V}_{qm}$, has not changed at the moment when purity of state has changed. Opposite to this is to assume that the diagonality of QM part of state, with respect to the basis which is privileged at that time, has not changed. Before the moment $t_o$, the QM part of state has been diagonal with respect to the eigenbasis of that observable for which $|\Psi(t_o)\rangle$ is the eigenstate. Only this basis can be characterized as privileged for that time because the corresponding observable has been used for preparation. For physics, each other basis, including the eigenbasis of $\hat{H}_{qm}$ and $\hat{V}_{qm}$, is less important, *i.e.*, their significance comes from mathematics, not from physics - they can be used just to express the same state in different manners. After the moment $t_o$ privileged basis is the eigenbasis of $\hat{V}_{qm}$ (and $\hat{H}_{qm}$) because this observable is measured. So, instead of claiming that the nondiagonality with respect to the basis which is going to become privileged should not change, one can claim that the diagonality with respect to the actually privileged basis should not change. These statements express two different types of reasoning: the first one concentrated on the formal aspect of the operators representing states (leading to (8)), while the other one concerned about the meaning (leading to (9)).

If the mentioned nondiagonality of QM part of initial state has survived $t_o$, then, according to (6), there would be the CM systems in (realizable) states $\hat{\rho}^{ij}_{cm}(t)$ coupled to the QM nondiagonal terms, as is given by (8). But, the probability of event $\hat{I} \otimes |q_{ij}(t)\rangle\langle q_{ij}(t)| \otimes |p_{ij}(t)\rangle\langle p_{ij}(t)|$ for the state (8) is equal to zero for all $t > t_o$, where $i \neq j$. Neither apparatus would be in any of the states $\hat{\rho}^{ij}_{cm}(t)$ with $i \neq j$ after the beginning of measurement. (This is not the case for $i = j$.) So, if the statements about probability are of any importance, before proclaiming (6) as unadequate for it does not save the non-negativity of initial state, one should accept that in the states $\hat{\rho}^{ij}_{cm}(t)$ $(i \neq j)$ neither apparatus would be. The consequence of this is that the assumption of survived QM nondiagonal terms is not correct. In physics, where the probability is a significant concept, just found is enough to conclude that (9) should be taken as solution. Simultaneously by finding that (8) is unphysical, one finds why it is so: it is unphysical because some states of CM systems that are not exhibited by any apparatus are kept in the representation of state of hybrid system. By taking this into account, *i.e.*, by reexpressing (8) with this in mind, one will find (9) as the proper state of hybrid system.

Finally, the validity of the hybrid systems dynamical equation can be verified on situations for which it is easy to say what behavior is desired. For example, the hybrid systems dynamical equation gives the standard one-to-one evolutions of QM and CM subsystems when the interaction term in Hamiltonian is absent. In this case evolved states are of the same purity and non-negativity as initial states. Moreover, for the above given Hamiltonian and the initial state of hybrid system $\sum_i |c_i(t_o)|^2 |\psi_i\rangle\langle\psi_i| \otimes |q_o\rangle\langle q_o| \otimes |p_o\rangle\langle p_o|$, evolved state is not unphysical, it is (9). These examples justify the hybrid systems dynamical equation as the proper one. So, it is likely that this holds for the case addressing the problem of measurement.

# 5 Concluding remarks

Without an operator formulation of classical mechanics, the analysis of the problem of measurement in the hybrid systems approach would not be complete. Firstly, this formulation enabled us to consider pure correlated state and then, after finding that such state cannot satisfy dynamical equation, to conclude that this dynamical equation produces noncausal evolution: when pure initial state of quantum system is not an eigenstate of the measured observable, initial state of hybrid system, which is also pure, necessarily and instantaneously transforms in some mixed correlated state. Secondly, when it was not so obvious how dynamical equation should be solved, the operator formulation offered support for one particular way.

The choice of a state representing hybrid system after the beginning of measurement is important since appropriateness of the HSA for physics depends on it. Both states that do satisfy dynamical equation for the given Hamiltonian are same regarding the impurity and absence of CM $i \neq j$ terms, so the essential part of physical meaning is one and the same. Only the way of expressing these differs from (8) to (9). For their properties, perhaps it would not be wrong to say that (9) is the physical result of hybrid systems dynamics and that (8) is a physically unacceptable mathematical solution.

The third usefulness of the operator formulation of classical mechanics is in that it allows one to design, let say, a dynamical model of instantaneous decoherence. Namely, in the resulting proposal of HSA, the partial derivations in the Poisson bracket change the CM nondiagonal terms at $t_o$ (if the initial state is seen as (7) with $t = t_o$) and then obstruct their further time development according to (10), *i.e.*, these derivations annihilate CM nondiagonal terms. So, in this proposal, the dynamics is the cause of collapse. The reduction of quantum mechanical state is the consequence of disappearance of classical mechanical $i \neq j$ terms. The part of interpretation of (8), which is meaningful from the point of view of everyday experience, has lead to the same conclusion: terms $\hat{\rho}^{ij}_{qm}(t)$ vanish because to them related and *per se* realizable events $\hat{\rho}^{ij}_{cm}(t)$ cannot occur. In another words, the reason for decoherence of QM state in case of a measurement lies in the Liouville equation. It is linear only in probability densities within the framework of commutative operators that represent position and momentum of classical systems, in difference to the Schrödinger equation which is linear in both: the probability densities and the probability amplitudes.

In almost the same manner as the action of projectors has described the measurement in standard quantum mechanics, the action of partial derivations do it here. If one compares the standard formulation of QM and the operator formulation of HSA, one finds them similar for they treat decoherence as instantaneous process. They differ since decoherence is dynamical here. The operator formulation of HSA in this way answers one question aroused in quantum mechanics: how the collapse should be described. But, there is another, more important question: why it happens. The hybrid system approach does not ask for some *ad hoc* concepts to explain the collapse of state; the non-negativity of probabilities is enough. Because of the non-negativity of probabilities, the collapse of state is the only possible way of evolution for physical systems in the considered case and it is as ordinary as the one-to-one evolutions are in other cases. If one wants to stay within the formulation of QM in one Hilbert space, then the HSA puts the projection postulate on more solid ground. It is not related to the consciousness of the observer, but to the non-negativity of probabilities.

The non-negativity of probabilities is, and should be, incorporated among the first principles of any physical theory. The hybrid system approach differs from classical and quantum mechanics only in that this principle should be invoked not just at the beginning, when the initial state is chosen, but for the moments at which states lose purity as well. This rule offers substitution of our search for a solution and it is not in contradiction with these two mechanics. There are no such moments when only Schrödinger or Liouville equation is solved within the Hilbert space and phase space, respectively, so there is no rule which would be contradicted. If it is represented (like some kind of superselection rule) in $\mathcal{H}_{qm} \otimes \mathcal{H}_{cm}^{q} \otimes \mathcal{H}_{cm}^{p}$ as a restriction to consider only states that are non-negative operators, then there would be only two possibilities for a correlated state in the analyzed case: the coherent mixture (7) and the noncoherent mixture (9). The state (9) would follow immediately after finding that (7) cannot satisfy the equation of motion. (There is strong similarity between this and the way of solving the Maxwell quations where only physicaly meaningful solution is retained.)

Roughly speaking, the procedure of solving differential equations consists in two steps. The first one is to find all functions that satisfy it (if there is any) and the second is, if there are more than one function, to select one by imposing some condition. The most often used is the Cauchy condition. Adapted to the present framework, it reads: the state at later times is the one which for $t = t_o$ becomes equal to the initial state. With this condition one wants to express assumed continuity of state. The state (8) obviously follows in this way and, since this state is unphysical, the HSA shows that the state of physical systems in considered case has to evolve discontinuously. From our point of view, this strongly recommends the HSA for a theory of combined classical and quantum systems.

The objections addressing the relevance of HSA for physics are closely related to the application of the Cauchy condition in, let say, careless manner. We believe that it is not correct to take it as the unique supplementary condition and that it is not appropriate to impose it without noticing that something dramatic happens with the initial state at the moment when evolution begins. If one would disregard the unavoidable change of purity of initial state treating it as unimportant, then one would go out of physics from the very beginning. Moreover, then one cannot discuss the physical meaning of solution at the end because it would make such consideration inconsistent. Only after finding that (according to the discussion based on (7) and (10)) the initial state has changed instantaneously and discontinuously, one should apply the Cauchy condition for then it is adequate because the further evolution is causal and in all aspects continuous. If this, the rule to invoke the non-negativity of probability for the moments at which states lose purity and (10) are new at all, these rules are the slightest possible modifications of the previously used ones. Or, perhaps, they are just the accommodation of standard rules to new situations.

Needless to say, the state (9) is in agreement with what is usually expected to happen when the problem of measurement is considered in an abstract and ideal form. To each state of the measured quantum system, which are the eigenstates of the measured observable, corresponds one pointer position and momentum. The $i$-th eigenvalue of measured observable occurs with probability $|c_i(t_o)|^2$ and, as was said, (9) takes place immediately after the apparatus in state $|q_o\rangle \otimes |p_o\rangle$ has started to measure $\hat{V}_{qm}$ on the system in pure state $|\Psi(t_o)\rangle$, which can be seen as $\sum_i c_i(t_o)|\psi_i\rangle$.

Once noticed, the departure from strict causality would also be noticed in (all) other aspects as some strange feature. For example, in [6] it was found that, so called, universal privileged times in dynamics of hybrid systems appear. Here, $t_o$ is such a moment. In contrast to opinion expressed there, we believe that this is a rather nice property of the approach. Namely, for the described process, and all other that can be treated in the same way, pure state can evolve into noncoherent mixture, while noncoherent mixture cannot evolve into coherent mixtures - pure states, *i.e.*, when the non-negativity of probability is respected, such processes are irreversible. This means that for them the entropy can only increase or stay constant. Then, the distinguished moments of the increase of entropy can be used for defining an arrow of time.

## References


[1] I. V. Aleksandrov, Z. Naturforsch. **36a** (1981) 902

[2] W. Boucher and J. Trashen, Phys. Rev. D Vol.**37** No.**4** (1988) 3522

[3] O. V. Prezhdo and V. V. Kisil, Phys. Rev. A Vol.**56** No.**1** (1997) 162

[4] A. Anderson, Phys. Rev. Lett. Vol.**74** No.**5** (1995) 621

[5] L. L. Salcedo, Phys. Rev. A Vol.**54** No.**4** (1996) 3657

[6] J. Caro and L. L. Salcedo: Obstructions to Mixing Classical and Quantum Dynamics, quant-ph/9812046

[7] A. Peres: Quantum Theory: Concepts and Methods, (Kluwer, 1993)

[8] J. G. Muga and R. F. Snider, Europhys. Lett., Vol.**19** No.**7** (1992) 569